RESEARCH STATEMENT

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Summary

My research interest is concerned with applied mathematics and nonlinear partial differential equations. Our work is focused Cauchy problems, typically with periodic on boundary conditions or in "the whole" $\mathbb{R}^n$. Typically well-posedness results hold a priori only for short time intervals. A basic problem is to establish whether such intervals can be taken of arbitrary length. When it is not the case, one expects to find a maximal existence time $T^* < \infty$ and some spatial norms of the solution such that $\|u(t,\cdot)\|$ is finite for $t \in (0, T^*)$ and becomes unbounded as $t \uparrow T^*$. In our works we address such global existence versus blowup issues. Our goal is to provide necessary or sufficient conditions (or both), for the initial data $u_0(x)$, guaranteeing that the lifespan time T^* of the solution arising from u_0 is finite or not.

We study two kinds of equations: nonlinear parabolic equations and a class of dispersive wave equations (including, e.g. Camassa-Holm, Degasperis-Procesi, shallow water, rod equation, b family of equations).

A model case: The nonlinear heat equation

As a model case for nonlinear heat equations we considered the following equation with cubic nonlinearity:

$$\begin{cases} \partial_t u = \Delta u + u^3 & x \in \mathbb{R}^3 \quad t \in [0,T] \\ u(0,x) = u_0(x), \end{cases} \tag{0.1}$$

where $0 < T \leq \infty$ and $u = u(x,t)$ is a real valued function of $x \in \mathbb{R}^3$ and $t \geq 0$. It is convenient to rewrite (0.1) in the equivalent integral formulation

$$u(t,x) = u(t) = e^{t\Delta}u_0(x) + \int_0^t e^{(t-\tau)\Delta}u^3(\tau,x)\,d\tau. \tag{0.2}$$

This equation shares with the Navier-Stokes equation the same translation and dilatation invariance, *i.e.*, if $u(x,t)$ is a solution of (0.1), or Navier-Stokes problem, so are $\lambda u(\lambda x, \lambda^2 t)$, with $\lambda > 0$ and $u(x - x_0, t - \tau)$ for $x_0 \in \mathbb{R}$, $\tau \geq 0$. The initial condition is modified accordingly. Our interest is to work in Banach spaces invariant with respect to this scaling. For the initial data, the only Lebesgue space invariant under this scaling is $L^3(\mathbb{R}^3)$. F. Weissler [20] showed that the Cauchy problem for (0.1) is locally well-posed in $L^p(\mathbb{R})$, for $p \geq 3$, while E. Terraneo [19] proved a non-uniqueness result for mild solutions of (0.1) which take values in $L^3(\mathbb{R}^3)$. We utilize the homogeneous Besov spaces $\dot{B}_q^{-\sigma,\infty}(\mathbb{R}^3) \subset L^3(\mathbb{R})$, where $3 < q < 9$, $\sigma = 1 - \frac{3}{q}$. Such homogeneous Besov spaces are invariant under this scaling. We first slightly extend a result of Y. Meyer in [14], showing that if u_0 is small enough in $\dot{B}_q^{-\sigma,\infty}(\mathbb{R}^3)$, then there is global existence of the solution $u(x,t)$ of (0.1) in $C([0,+\infty[;\dot{B}_q^{-\sigma,\infty}(\mathbb{R}^3))$. This relies on the application of Kato's pertubative method [11]. Conversely, inspired by the article of Mongomery-Smith [15], we prove ill-posedness for $\dot{B}_q^{-\sigma,\infty}(\mathbb{R}^3)$, where $q>9$. Also we show norm inflation phenomena of the solution for some initial data, more precisely, we prove that for all $\delta>0$ there exists an initial data u_0 with $\|u_0\|_{B^{-1}_{\infty,\infty}}\leq\delta$ such that the corresponding solution u satisfies $\|u(t)\|_{B^{-1}_{\infty,\infty}}\geq 1/\delta$, for some $t<\delta$. This is in the same spirit as Bourgain and Pavlović [4] on norm inflation for solutions of the Navier–Stokes equation.

On permanent and breaking waves in hyperelastic rods and rings

In this part, we work on a class of dispersive wave equations. In particular, we study the solutions of the Cauchy problem for the periodic rod equation written as follows:

$$\begin{cases} u_t+\gamma u u_x=-\partial_x p*\left(\dfrac{3-\gamma}{2}\,u^2+\dfrac{\gamma}{2}\,u_x^2\right), & t\in(0,T),\ x\in\mathbb{S},\\ u(0,x)=u_0(x). \end{cases} \tag{0.3}$$

with $\mathbb{S}=\mathbb{R}/\mathbb{Z}$, the unit circle. The function p in (0.3) is the kernel of the convolution operator $(1-\partial_x^2)^{-1}$. It is the continuous 1-periodic function given by

$$p(x)=\frac{\cosh(x-[x]-1/2)}{2\sinh(1/2)}, \tag{0.4}$$

where $[\cdot]$ denotes the integer part. A first physical motivation comes from the study of the response of hyper-elastic rings under the action of an initial radial stretch. As the nonlinear dispersive waves propagating inside it could eventually lead to cracks, an important problem is the determination of conditions that must be fulfilled in order to prevent their formation. A second reason for studying periodic solutions is that periodic waves spontaneously arise also in hyper-elastic rods: indeed, it has been recently observed that the solitary waves propagating inside an ideally infinite length rod can suddenly feature a transition into waves with finite period as their amplitude increases, see [7]. Our third motivation comes from the study of shallow water waves inside channels. Indeed, the Camassa–Holm equation (at least in the dispertionless case) is a particular case, corresponding to $\gamma=1$, of the rod equation above: if the motion of small amplitude waves is usually modeled by the KdV equation, larger amplitude waves, and in particular breaking waves, are more accurately described by the Camassa–Holm equation. In fact, both the KdV and the Camassa–Holm equation can be rigorously derived as an asymptotic model from the free surface Euler equations for irrotational inviscid flows.

In [2], We prove that the only global strong solution of the periodic rod equation vanishing in at least one point $(t_0,x_0)\in\mathbb{R}^+\times\mathbb{S}$ is the identically zero solution. Our analysis relies on the application of new local-in-space blowup criteria: we establish that if $|\gamma|$ is *not too small*, then there exists a constant $\beta_\gamma>0$ such that if

$$u_0'(x_0)>\beta_\gamma|u_0(x_0)| \quad \text{if } \gamma<0, \qquad \text{or} \qquad u_0'(x_0)<-\beta_\gamma|u_0(x_0)| \quad \text{if } \gamma>0 \tag{0.5}$$

in at least one point $x_0 \in \mathbb{S}$, then the solution arising from $u_0 \in H^s(\mathbb{S})$ must blow up in finite time. More precisely, the following upper bound estimate for T^* holds,

$$T^* \leq \frac{2}{\gamma\sqrt{u_0'(x_0)^2 - \beta_\gamma^2 u_0(x_0)^2}} \tag{0.6}$$

and, for some $x(t) \in \mathbb{S}$, the blow up rate is

$$u_x(t, x(t)) \sim -\frac{2}{\gamma(T^* - t)} \qquad \text{as } t \to T^*.$$

An analogue but weaker result was recently established in a previous paper [1], that dealt with non-periodic solutions on the whole real line with vanishing boundary conditions as $x \to \infty$.

Blowup for the generalized hyper-elastic rod equation

The next equation that we addressed is the nonlinear dispersive wave equation on the real line,

$$\begin{cases} u_t + f'(u)u_x + \partial_x p * \left[g(u) + \frac{f''(u)}{2}u_x^2\right] = 0, & x \in \mathbb{R}, \quad t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{0.7}$$

Here $p(x) = \frac{1}{2}e^{-|x|}$. The function p in (0.7) is the kernel of the convolution operator $(1-\partial_x^2)^{-1}$ on the real line. For appropriate choices of the functions f and g, Equation (0.7) includes well known models, such as Dai's equation for the study of vibrations inside elastic rods or the Camassa–Holm equation modelling water wave propagation in shallow water. When $f(u) = \frac{u^{Q+1}}{Q+1}$ and $g(u) = \kappa u + \frac{Q^2+3Q}{2(Q+1)}u^{Q+1}$ one recovers from (0.7) another class of equations with interesting mathematical properties, studied in [10]. In [3], our purpose is to establish a new blowup criterion for equation (0.7), that is both more natural and more general than earlier blowup criteria. In particular we can handle more general boundary conditions, encompassing also the case of solutions not necessarily vanishing at infinity. We are also able to cover the case $f(u) = u^2$ and $g(u) = \kappa u + u^2$(Camassa-Holm equation with dispersion). Contrary to previously known blowup criteria, like those in [6, 13, 21], our criterion has the specific feature of being *purely local* in the space variable: indeed our blowup condition only involves the values of $u_0(x_0)$ and $u_0'(x_0)$ in a single point x_0 of the real line. Under appropriate conditions on the functions f and g, provided the initial datum $u_0 \in H^s$ $(s > 3/2)$, satisfies

$$\exists\, x_0 \in \mathbb{R} \quad \text{such that} \quad u_0'(x_0) < -\beta\big|u_0(x_0) - c\big|,$$

where β and c are two real constants depending on the shape of the functions f and g, then the solution arising from $u_0 \in H^s(\mathbb{R})$ must blow up in finite time. In this case we get an estimate of the blowup time of the form:

$$T^* \leq \frac{4}{\gamma\sqrt{4u_0'(x_0)^2 - \left(\sqrt{1 \pm 8K^2} - 1\right)^2\big(u_0(x_0) - c\big)^2}}. \tag{0.8}$$

Blowup for b-Family of Equations

A last issue that we considered is the Cauchy problem for the class of b-family equations.

$$\begin{cases} u_t + uu_x + \partial_x p * \left[\frac{b}{2}u^2 + \left(\frac{3-b}{2}\right)u_x^2\right] = 0, & x \in \mathbb{S}, \quad t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{S} \\ u(t,x) = u(t,x+1) \quad t \geq 0, \end{cases} \tag{0.9}$$

Here $p(x)$ is as in (0.4). Here b is a real parameter, and $u(x,t)$ is the velocity of the fluid on the torus $\mathbb{S}$. The b-family equation can be derived as a family of asymptotically equivalent shallow water wave equations that emerges at quadratic-order accuracy for any $b \neq 1$ by an appropriate Kodama transformation [8, 9]. Again, when $b = 2$ and $b = 3$, (0.9) became (C-H) and (D-P) respectively. These values are the only values for which (0.9) is completely integrable. The Cauchy problem for the b-family equations is locally well posed in the Sobolev space H^s for any $s > \frac{3}{2}$, see [16, 18]. In fact, it is proved in [18], that maximal lifespan of the solution of (0.9) may be chosen independently of s. We focus on blow-up criteria as well as on estimates about the lifespan of the solution. The blow-up criteria are usually non-local with respect to space. Thus our contribution provides a sufficient, local condition for the lifespan to be bounded. Our result reads as follows: Let $1.0012 \approx \alpha_0 \leq b \leq 3$. There is $\beta_b > 0$, such that if $u_0 \in H^s(\mathbb{S})$, with $s > \frac{3}{2}$, satisfies

$$u_0'(x_0) < -\beta_b \, |u_0(x_0)| \tag{0.10}$$

in at least one point $x_0 \in \mathbb{S}$, then the solution arising $u_0 \in H^s(\mathbb{S})$ must blow-up in finite time. Notice that in earlier papers blowup results involved more stringent conditions on the parameter b. Our technical restriction $b \geq \alpha_0$ is indeed very close to the expected physical condition $b > 1$. As in the previous section, we provide explicit estimates on T^*. Our analysis includes estimates and a numerical computations of β_b.

1. Plans for future research

We present two research problems. The first one is closely related to a conjecture on the well–posedness of the cubic heat equation inspired by our previous work. The second is a work in progress.

Well-posedness for the cubic heat equation.

(1) With respect to equation (0.1), consider a Banach space X such that $\mathcal{S}(\mathbb{R}^3) \subset X \subset \mathcal{S}'(\mathbb{R}^3)$ with continous injections. We suppose that $\|\cdot\|_X$ is invariant with respect to the same scaling of $L^3(\mathbb{R}^3)$. Also we assume that $u_0 \in \mathcal{S}(\mathbb{R}^3)$, such that $\|u_0\|_X < \epsilon$. By Meyer [14] and our prolongation, we know that if $X = L^3(\mathbb{R}^3)$, or $X = \dot{B}_q^{-\sigma,\infty}(\mathbb{R}^3)$, where $3 < q < 9$, $\sigma = 1 - \frac{3}{9}$ and if ϵ is sufficiently small then the Cauchy problem, then (0.1) admits a global solution. What is the largest possible scaling invariant space X where the same conclusion does hold ?

Notice that in the case of the Navier-Stokes equations the answer to this question is known, and the maximal space is provided by the Koch-Tataru space [12]. But the problem for the cubic heat equation is still open. In our research, we have two partial conclusions. If this maximal Banach space X does exist, then it must verify the following assertions:

- $\bigcup_{3<q<9} \dot{B}_q^{-\sigma,\infty}(\mathbb{R}^3) \subset X$.
- $\dot{B}_9^{-\frac{2}{3},\infty}(\mathbb{R}^3) \not\subseteq X$.

We conjecture that the maximal space X can be defined by suitably modifying the Koch-Tataru space in order to include L^3_{loc}-functions. However, at the moment we do not know how to prove the relevant tri-linear estimates in such space.

(2) When $u_0 \in \dot{B}_q^{-\sigma,\infty}$, with $q > 9$ or $\dot{B}_q^{-\frac{2}{3},k}$, with $k > 3$, the solution of (0.1) arising from u_0, blows up in finite time. Observing the Sobolev embeddings

$$L^3 = \dot{B}_3^{0,3} \subset \dot{B}_9^{-\frac{2}{3},3} \subset \dot{B}_9^{-\frac{2}{3},\infty},$$

another interesting open problem appears: is it possible to get the global well-posedness putting a smallness condition on u_0 in the space $X = \dot{B}_9^{-\frac{2}{3},\infty}$? One might speculate that in $\dot{B}_9^{-\sigma,\infty}$ we have ill-posedness and try to build u_0 such that, no matter how small is u_0 in $\dot{B}_9^{-\frac{2}{3},\infty}$, the solution u arising of u_0 blows up.

Periodic waves i generalized rods In (0.7), we did not investigate the periodic case. It seems hopeful that adapting our methods could be effective in the case of the torus $\mathbb{S}^1$. In fact, we expect that in the periodic case the results cold be stronger than in the case of the whole real line.

New challenges. I intend to continue studying Nonlinear Analysis and in particular in incompressible fluids. Here are a few of the questions on which I would like to work in the near future.

- With Professor Alvarez Samaniego, we are investigating the following Cauchy problem:

$$v_t + vv_x + v_{xxx} + \eta(\mathcal{H}v_x + \mathcal{H}v_{xxx}) = 0, \quad v(\cdot,0) = \phi(\cdot), \tag{1.1}$$

where $\mathcal{H}$ denote the Hilbert transform and $\eta \geq 0$. The equation (1.1) was proposed by Ostrovsky to describe the radiational instability of long waves in a stratified shear flow [17]. Mr. Alvarez in his Thesis, developed the local and global theory in H^s, with $s > 1$. He also got results in the space $\mathcal{F}_{r,s}(\mathbb{R}) = H^s(\mathbb{R}) \cap L^2_r(\mathbb{R})$, where L^2_r is a weighted L^2-space. We would like to obtain similar results in the Besov spaces of the form $B^s_{2,r}(\mathbb{R})$. The motivation is to improve on the regularity index s. Moreover, we would like to better describe the persistence properties of solutions in weighted spaces, in order to have a sharp description on the behavior of solutions at the spatial infinity.

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